

Occupation densities for certain processes related to fractional Brownian motion

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Abstract

In this paper we establish the existence of a square integrable occupation density for two classes of stochastic processes. First we consider a Gaussian process with an absolutely continuous random drift, and secondly we handle the case of a (Skorohod) integral with respect to the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. The proof of these results uses a general criterion for the existence of a square integrable local time, which is based on the techniques of Malliavin calculus.

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1 Introduction

Local times for semimartingales have been widely studied. See for example the monograph [11] and the references therein. On the other hand, local times of Gaussian processes have also been the object of a rich probabilistic literature; see for example the recent paper [8] by Marcus and Rosen. A general criterion for the existence of a local time for a wide class of anticipating processes, which are not semimartingales or Gaussian processes, was established by Imkeller and Nualart in [6]. The proof of this result combines the techniques of Malliavin calculus with the criterion given by Geman and Horowitz in [4]. This criterion was applied in [6] to the Brownian motion with an anticipating drift, and to indefinite Skorohod integral processes.

The aim of this paper is to establish the existence of the occupations densities for two classes of stochastic processes related to the fractional Brownian motion, using the approach introduced in [6]. First we consider a Gaussian process $B = \{B_t, t \in [0, 1]\}$ with an absolutely continuous random drift

$$X_t = B_t + \int_0^t u_s ds,$$

where u is a stochastic process measurable with respect to the σ -field generated by B . We assume that the variance of the increment of the Gaussian process B on an interval $[s, t]$ behaves as $|t - s|^{2\rho}$, for some $\rho \in (0, 1)$. This includes, for instance, the bifractional Brownian motion with parameters $H, K \in (0, 1)$. Under reasonable regularity hypotheses imposed to the process u we prove the existence of a square integrable occupation density with respect to the Lebesgue measure for the process X .

Our second example is represented by the indefinite divergence (Skorohod) integral $X = \{X_t, t \in [(0, 1)]\}$ with respect to the fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, that is

$$X_t = \int_0^t u_s \delta B_s^H.$$

We provide integrability conditions on the integrand u and its iterated derivatives in the sense of Malliavin calculus in order to deduce the existence of a square integrable occupation densities for X .

We organized our paper as follows. Section 2 contains some preliminaries on the Malliavin calculus with respect to Gaussian processes. In Section 3 we prove the existence of the occupation densities for perturbed Gaussian processes and in Section 4 we treat the case of indefinite divergence integral processes with respect to the fractional Brownian motion.

2 Preliminaries

Let $\{B_t, t \in [0, 1]\}$ be a centered Gaussian process with covariance function

$$R(t, s) := E(B_t B_s),$$

defined in a complete probability space (Ω, \mathcal{F}, P) . By \mathcal{H} we denote the canonical Hilbert space associated to B defined as the closure of the linear space generated by the indicator functions $\{\mathbf{1}_{[0,t]}, t \in [0, 1]\}$ with respect to the inner product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R(t, s), \quad s, t \in [0, 1].$$

The mapping $\mathbf{1}_{[0,t]} \rightarrow X_t$ can be extended to an isometry between \mathcal{H} and the first Gaussian chaos generated by B . We denote by $B(\varphi)$ the image of an element $\varphi \in \mathcal{H}$ by this isometry.

We will first introduce some elements of the Malliavin calculus associated with B . We refer to [10] for a detailed account of these notions. For a smooth random variable $F = f(B(\varphi_1), \dots, B(\varphi_n))$, with $\varphi_i \in \mathcal{H}$ and $f \in C_b^\infty(R^n)$ (f and all its partial derivatives are bounded) the derivative of F with respect to B is defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B(\varphi_1), \dots, B(\varphi_n)) \varphi_j.$$

For any integer $k \geq 1$ and any real number $p \geq 1$ we denote by $\mathbb{D}^{k,p}$ the Sobolev space defined as the the closure of the space of smooth random variables with respect to the norm

$$\|F\|_{k,p}^p = E(|F|^p) + \sum_{j=1}^k \|D^j F\|_{L^p(\Omega; \mathcal{H}^{\otimes j})}^p.$$

Similarly, for a given Hilbert space V we can define Sobolev spaces of V -valued random variables $\mathbb{D}^{k,p}(V)$.

Consider the adjoint δ of D in L^2 . Its domain is the class of elements $u \in L^2(\Omega; \mathcal{H})$ such that

$$E(\langle DF, u \rangle_{\mathcal{H}}) \leq C \|F\|_2,$$

for any $F \in \mathbb{D}^{1,2}$, and $\delta(u)$ is the element of $L^2(\Omega)$ given by

$$E(\delta(u)F) = E(\langle DF, u \rangle_{\mathcal{H}})$$

for any $F \in \mathbb{D}^{1,2}$. We will make use of the notation $\delta(u) = \int_0^1 u_s \delta B_s$. It is well-known that $\mathbb{D}^{1,2}(\mathcal{H})$ is included in the domain of δ . Note that $E(\delta(u)) = 0$ and the variance of $\delta(u)$ is given by

$$E(\delta(u)^2) = E(\|u\|_{\mathcal{H}}^2) + E(\langle Du, (Du)^* \rangle_{\mathcal{H} \otimes \mathcal{H}}), \quad (2.1)$$

if $u \in \mathbb{D}^{1,2}(\mathcal{H})$, where $(Du)^*$ is the adjoint of Du in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$. We have Meyer's inequality

$$E(|\delta(u)^p|) \leq C_p (E(\|u\|_{\mathcal{H}}^p) + E(\|Du\|_{\mathcal{H} \otimes \mathcal{H}}^p)), \quad (2.2)$$

for any $p > 1$. We will make use of the property

$$F\delta(u) = \delta(Fu) + \langle DF, u \rangle_{\mathcal{H}}. \quad (2.3)$$

if $F \in \mathbb{D}^{1,2}$ and $u \in \text{Dom}(\delta)$ such that $Fu \in \text{Dom}(\delta)$. We also need the commutativity relationship between D and δ

$$D\delta(u) = u + \int_0^1 Du_s \delta B_s, \quad (2.4)$$

if $u \in \mathbb{D}^{1,2}(\mathcal{H})$ and the process $\{D_s u, s \in [0, 1]\}$ belongs to the domain of δ .

Throughout this paper we will assume that the centered Gaussian process $B = \{B_t, t \in [0, 1]\}$ satisfies

$$C_1(t-s)^{2\rho} \leq E(|B_t - B_s|^2) \leq C_2(t-s)^{2\rho}, \quad (2.5)$$

for some $\rho \in (0, 1)$ with C_1, C_2 two positive constants not depending on t, s . It will follow from the Kolmogorov criterium that B admits a Hölder continuous version of order δ for any $\delta < \rho$.

Throughout this paper we will denote by C a generic constant that may be different from line to line.

Example 1 *The bifractional Brownian motion (see, for instance [5]), denoted by $B^{H,K}$, is defined as a centered Gaussian process starting from zero with covariance*

$$R(t, s) = \frac{1}{2^K} \left((t^{2H} + s^{2H})^K - |t - s|^{2HK} \right) \quad (2.6)$$

where $H \in (0, 1)$ and $K \in (0, 1]$. When $K = 1$, then we have a standard fractional Brownian motion denoted by B^H . It has been proven in [5] that for all $s \leq t$,

$$2^{-K}|t - s|^{2HK} \leq E \left| B_t^{H,K} - B_s^{H,K} \right|^2 \leq 2^{1-K}|t - s|^{2HK} \quad (2.7)$$

so relation (2.5) holds with $\rho = HK$. A stochastic analysis for this process can be found in [7] and a study of its occupation densities has been done in [2], [12].

For a measurable function $x : [0, 1] \rightarrow \mathbb{R}$ we define the occupation measure

$$\mu(x)(C) = \int_0^1 \mathbf{1}_C(x_s) ds,$$

where C is a Borel subset of \mathbb{R} and we will say that x has an occupation density with respect to the Lebesgue measure λ if the measure μ is absolutely continuous with respect to λ . The occupation density of the function x will be the derivative $\frac{d\mu}{d\lambda}$. For a continuous process $\{X_t, t \in [0, 1]\}$ we will say that X has an occupation density on $[0, 1]$ if for almost all $\omega \in \Omega$, $X(\omega)$ has an occupation density on $[0, 1]$.

We will use the following criterium for the existence of occupation densities (see [6]). Set $T = \{(s, t) \in [0, 1]^2 : s < t\}$.

Theorem 1 *Let $\{X_t, t \in [0, 1]\}$ be a continuous stochastic process such that $X_t \in \mathbb{D}^{2,2}$ for every $t \in [0, 1]$. Suppose that there exists a sequence of random variables $\{F_n, n \geq 1\}$ with $\bigcup_n \{F_n \neq 0\} = \Omega$ a.s. and $F_n \in \mathbb{D}^{1,1}$ for every $n \geq 1$, two sequences $\alpha_n > 0, \delta_n > 0$, a measurable bounded function $\gamma : [0, 1] \rightarrow \mathbb{R}$, and a constant $\theta > 0$, such that:*

a) *For every $n \geq 1$, $|t - s| \leq \delta_n$, and on $\{F_n \neq 0\}$ we have*

$$\langle \gamma D(X_t - X_s), \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}} > \alpha_n |t - s|^\theta, \quad \text{a.s..} \quad (2.8)$$

b) *For every $n \geq 1$*

$$\int_T E(\langle \gamma D F_n, \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}}) |t - s|^{-\theta} dt ds < \infty. \quad (2.9)$$

c) *For every $n \geq 1$*

$$\int_T E \left(\left| F_n \left\langle \gamma^{\otimes 2} DD(X_t - X_s), \mathbf{1}_{[s,t]}^{\otimes 2} \right\rangle_{\mathcal{H}^{\otimes 2}} \right| \right) |t - s|^{-2\theta} ds dt < \infty. \quad (2.10)$$

Then the process $\{X_t, t \in [0, 1]\}$ admits a square integrable occupation density on $[0, 1]$.

Remark 1 *The original result has been stated in [6] with $\theta = 1$ in the case of the standard Brownian motion. On the other hand, by applying Proposition 2.3 and Theorem 2.1 in [6] it follows easily that this criterium can be stated for any $\theta > 0$.*

3 Occupation density for Gaussian processes with random drift

We study in this part the existence of the occupation density for Gaussian processes perturbed by a absolute continuous random drift. The main result of this section is the following.

Theorem 2 *Let $\{B_t, t \in [0, 1]\}$ be a Gaussian process satisfying (2.5). Consider the process $\{X_t, t \in [0, 1]\}$ given by*

$$X_t = B_t + \int_0^t u_s ds,$$

and suppose that the process u satisfies the following conditions:

1. $u \in \mathbb{D}^{2,2}(L^2([0, 1]))$.
2. $E \left(\left(\int_0^1 \|D^2 u_t\|_{\mathcal{H} \otimes \mathcal{H}}^p dt \right)^{q/p} \right) < \infty$, for some $q > 1$, $p > \frac{1}{1-\rho}$.

Then, the process X has a square integrable occupation density on the interval $[0, 1]$.

Proof: We are going to apply Theorem 1. Notice first that $X_t \in \mathbb{D}^{2,2}$ for all $t \in [0, 1]$. For any $0 \leq s < t \leq 1$, using (2.4) and (2.5) we have

$$\begin{aligned} \langle D(X_t - X_s), \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}} &= \langle \mathbf{1}_{[s,t]}, \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}} + \left\langle \int_s^t Du_r dr, \mathbf{1}_{[s,t]} \right\rangle_{\mathcal{H}} \\ &\geq C_1(t-s)^{2\rho} - \left| \left\langle \int_s^t Du_r dr, \mathbf{1}_{[s,t]} \right\rangle_{\mathcal{H}} \right| \\ &\geq C_1(t-s)^{2\rho} - \sqrt{C_2}(t-s)^\rho \int_s^t \|Du_r\|_{\mathcal{H}} dr. \end{aligned}$$

By Hölder's inequality, if $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\int_s^t \|Du_r\|_{\mathcal{H}} dr \leq (t-s)^{\frac{1}{q}} \left(\int_0^1 \|Du_r\|_{\mathcal{H}}^p dr \right)^{\frac{1}{p}}.$$

Fix a natural number $n \geq 2$, and choose a function $\varphi_n(x)$, which is infinitely differentiable with compact support, such that $\varphi_n(x) = 1$ if $|x| \leq n-1$, and $\varphi_n(x) = 0$, if

$|x| \geq n$. Set $F_n = \varphi_n \left(\left(\int_0^1 \|Du_t\|_{\mathcal{H}}^p dt \right)^{\frac{1}{p}} \right)$. The random variable F_n belongs to $\mathbb{D}^{1,q}$.

In fact, it suffices to write $F_n = \varphi_n(G)$, where

$$G = \sup_{\substack{h \in L^q([0,1]; \mathcal{H}) \\ \|h\| \leq 1}} \int_0^1 \langle Du_r, h_r \rangle_{\mathcal{H}} dr,$$

which implies

$$\begin{aligned} \|DF_n\|_{\mathcal{H}} &= \|\varphi'_n(G)DG\|_{\mathcal{H}} \leq \|\varphi'_n\|_{\infty} \sup_{\substack{h \in L^q([0,1]; \mathcal{H}) \\ \|h\| \leq 1}} \left\| \int_0^1 \langle D^2 u_r, h_r \rangle_{\mathcal{H}^{\otimes 2}} dr \right\|_{\mathcal{H}} \\ &\leq \|\varphi'_n\|_{\infty} \left(\int_0^1 \|D^2 u_r\|_{\mathcal{H}^{\otimes 2}}^p dr \right)^{\frac{1}{p}} \in L^q(\Omega). \end{aligned}$$

Then, on the set $\{F_n \neq 0\}$, $\left(\int_0^1 \|Du_t\|_{\mathcal{H}}^p dt \right)^{\frac{1}{p}} \leq n$, and we get

$$\begin{aligned} \langle D(X_t - X_s), \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}} &\geq C_1(t-s)^{2\rho} - n\sqrt{C_2}(t-s)^{\rho+\frac{1}{q}} \\ &= (t-s)^{2\rho} \left[C_1 - n\sqrt{C_2}(t-s)^{\frac{1}{q}-\rho} \right], \end{aligned}$$

and property a) of Theorem 1 holds with a suitable choice of α_n and δ_n because $\frac{1}{q} - \rho > 0$, and with $\theta = 2\rho$ and $\gamma = 1$.

Finally, conditions b) and c) can also be checked:

$$\int_T \frac{E \left(\left| \langle DF_n, \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}} \right| \right)}{|t-s|^{2\rho}} ds dt \leq \sqrt{C_2} \int_T \frac{E(\|DF_n\|_{\mathcal{H}})}{|t-s|^{\rho}} ds dt < \infty,$$

and

$$\begin{aligned} &\int_T \frac{E \left(\left| F_n \langle D^2(X_t - X_s), \mathbf{1}_{[s,t]}^{\otimes 2} \rangle_{\mathcal{H}^{\otimes 2}} \right| \right)}{|t-s|^{4\rho}} ds dt \\ &\leq \|F_n\|_{\infty} C_2 \int_T \frac{E(\|D^2(X_t - X_s)\|_{\mathcal{H}^{\otimes 2}})}{|t-s|^{2\rho}} ds dt < \infty, \end{aligned}$$

because

$$\begin{aligned} E(\|D^2(X_t - X_s)\|_{\mathcal{H}^{\otimes 2}}) &= E \left(\left\| \int_s^t D^2 u_r dr \right\|_{\mathcal{H}^{\otimes 2}} \right) \leq \int_s^t E(\|D^2 u_r\|_{\mathcal{H}^{\otimes 2}}) dr \\ &\leq (t-s)^{\frac{1}{q}} E \left[\left(\int_0^1 \|D^2 u_r\|_{\mathcal{H}^{\otimes 2}}^p dr \right)^{\frac{1}{p}} \right], \end{aligned}$$

and $\frac{1}{q} - 2\rho = 1 - \frac{1}{p} - 2\rho > -1$, because $p > \frac{1}{2(1-\rho)}$. ■

Remark 2 *These conditions are intrinsic and they do not depend on the structure of the Hilbert space \mathcal{H} . In the case of the Brownian motion, this result is slightly weaker than Theorem 3.1 in [6], because we require a little more integrability.*

4 Occupation density for Skorohod integrals with respect to the fractional Brownian motion

We study here the existence of occupation densities for indefinite divergence integrals with respect to the fractional Brownian motion. Consider a process of the form $X_t = \int_0^t u_s \delta B_s^H$, $t \in [0, 1]$, where B is fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, and u is an element of $\mathbb{D}^{1,2}(L^2([0, 1])) \subset \text{Dom}(\delta)$.

We know that the covariance of the fractional Brownian motion can be written as

$$E(B_t^H B_s^H) = \int_0^t \int_0^s \phi(\alpha, \beta) d\alpha d\beta, \quad (4.1)$$

where $\phi(\alpha, \beta) = H(2H - 1)|\alpha - \beta|^{2H-2}$. For any $0 \leq s < t \leq 1$, and $\alpha \in [0, 1]$ we set

$$f_{s,t}(\alpha) := \int_s^t \phi(\alpha, \beta) d\alpha d\beta. \quad (4.2)$$

We also know (see e.g. [10]) that the canonical Hilbert space associated to B satisfies:

$$L^2([0, 1]) \subset L^{\frac{1}{H}}([0, 1]) \subset \mathcal{H}. \quad (4.3)$$

The following is the main result of this section.

Theorem 3 *Consider the stochastic process $X_t = \int_0^t u_s \delta B_s^H$ where the integrand u satisfy the following conditions for some $q > \frac{2H}{1-H}$ and $p > 1$ such that $\frac{1}{p} + 2 < H(p+1)$:*

I1) $u \in \mathbb{D}^{3,2}(L^2([0, 1]))$.

I2) $\int_0^1 \int_0^1 [E(|D_t u_s|^p) + E(\|D_t D u_s\|_{\mathcal{H}}^p) + E(\|D_t D D u_s\|_{\mathcal{H} \otimes \mathcal{H}}^p)] ds dt < \infty$.

I3) $\int_0^1 E\left(|u_t|^{-\frac{p}{p-1}(q+1)}\right) dt < \infty$.

Then the process $\{X_t, t \in [0, 1]\}$ admits a square integrable occupation density on $[0, 1]$.

Proof: We will use the criteria given in [6] and recalled in Theorem 1. Condition I1) implies that $X_t \in \mathbb{D}^{2,2}$ for all $t \in [0, 1]$. On the other hand, from Theorem 7.8 in [7] (or also by a slightly modification of Theorem 5 in [1]) we obtain the continuity of the paths of the process X . Note that from Lemma 2.2 in [6] corroborated with hypothesis I3). we obtain the existence of a function $\gamma : [0, 1] \rightarrow \{-1, 1\}$ such that $\gamma_t u_t = |u_t|$ for almost all t and ω .

We are going to show conditions a), b) and c) of Theorem 1.

Proof of condition a): Fix $0 \leq s < t \leq 1$. From (2.4) we obtain

$$D(X_t - X_s) = u \mathbf{1}_{[s,t]} + \int_s^t Du_r \delta B_r^H,$$

and we can write

$$\langle \gamma(X_t - X_s), \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}} = \langle |u| \mathbf{1}_{[s,t]}, \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}} + \langle \gamma \int_s^t Du_r \delta B_r^H, \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}}. \quad (4.4)$$

We first study the term

$$\langle |u| \mathbf{1}_{[s,t]}, \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}} = \int_s^t \int_s^t |u_\alpha| \phi(\alpha, \beta) d\alpha d\beta = \int_s^t |u_\alpha| f_{s,t}(\alpha) d\alpha.$$

For any $q > 1$ we have

$$\begin{aligned} E(|B_t^H - B_s^H|^2) &= \int_s^t f_{s,t}(\alpha) d\alpha \\ &= \int_s^t (|u_\alpha| f_{s,t}(\alpha))^{\frac{q}{q+1}} (|u_\alpha| f_{s,t}(\alpha))^{-\frac{q}{q+1}} f_{s,t}(\alpha) d\alpha, \end{aligned}$$

and using Hölder's inequality with orders $\frac{q+1}{q}$ and $q+1$, we obtain

$$E(|B_t^H - B_s^H|^2) \leq \left(\int_s^t |u_\alpha| f_{s,t}(\alpha) d\alpha \right)^{\frac{q}{q+1}} \left(\int_s^t |u_\alpha|^{-q} f_{s,t}(\alpha) d\alpha \right)^{\frac{1}{q+1}}.$$

Hence, using that

$$f_{s,t}(\alpha) \leq f_{0,1}(\alpha) = H(2H-1) \int_0^1 |\alpha - \beta|^{2H-2} d\beta = H(\alpha^{2H-1} + (1-\alpha)^{2H-1}) \leq H,$$

we get

$$\int_s^t |u_\alpha| f_{s,t}(\alpha) d\alpha \geq C|t-s|^{\frac{2H(q+1)}{q}} Z_q^{-\frac{1}{q}}, \quad (4.5)$$

where $Z_q = \int_0^1 |u_\alpha|^{-q} d\alpha$.

On the other hand, for the second summand in the right-hand side of (4.4) we can write, using Hölder's inequality.

$$\begin{aligned} \left| \left\langle \gamma \int_s^t Du_r \delta B_r^H, \mathbf{1}_{[s,t]} \right\rangle_{\mathcal{H}} \right| &\leq \int_0^1 \left| \int_s^t D_\alpha u_r \delta B_r^H \right| f_{s,t}(\alpha) d\alpha \\ &\leq \left(\int_0^1 f_{s,t}(\alpha)^{\frac{p}{p-1}} d\alpha \right)^{\frac{p-1}{p}} \\ &\quad \times \left(\int_0^1 \left| \int_s^t D_\alpha u_r \delta B_r \right|^p d\alpha \right)^{\frac{1}{p}}. \end{aligned} \quad (4.6)$$

We can write

$$\begin{aligned} \left(\int_0^1 f_{s,t}(\alpha)^{\frac{p}{p-1}} d\alpha \right)^{\frac{p-1}{p}} &= c_H \left\| \int_s^t |\cdot - \beta|^{2H-2} d\beta \right\|_{L^{\frac{p}{p-1}}([0,1])} \\ &\leq c_H \left\| \mathbf{1}_{[s,t]} * |\cdot|^{2H-2} \mathbf{1}_{[-1,1]} \right\|_{L^{\frac{p}{p-1}}(\mathbb{R})}, \end{aligned} \quad (4.7)$$

where $c_H = H(2H-1)$. Young's inequality with exponents a and b in $(1, \infty)$ such that $\frac{1}{a} + \frac{1}{b} = 2 - \frac{1}{p}$ yields

$$\left\| \mathbf{1}_{[s,t]} * |\cdot|^{2H-2} \mathbf{1}_{[-1,1]} \right\|_{L^{\frac{p}{p-1}}(\mathbb{R})} \leq \left\| \mathbf{1}_{[s,t]} \right\|_{L^a(\mathbb{R})} \left\| |\cdot|^{2H-2} \mathbf{1}_{[-1,1]} \right\|_{L^b(\mathbb{R})}. \quad (4.8)$$

Choosing $b < \frac{1}{2-2H}$ and letting $\eta = \frac{1}{a} < 2H - \frac{1}{p}$ we obtain from (4.6), (4.7), and (4.8)

$$\left| \left\langle \gamma \int_s^t Du_r \delta B_r^H, \mathbf{1}_{[s,t]} \right\rangle_{\mathcal{H}} \right| \leq C |t-s|^\eta \left(\int_0^1 \left| \int_s^t D_\alpha u_r \delta B_r \right|^p d\alpha \right)^{\frac{1}{p}}.$$

Now we will apply Garsia-Rodemich-Ramsey's lemma (see [3]) with $\Phi(x) = x^p$, $p(x) = x^{\frac{m+2}{p}}$ and to the continuous function $u_s = \int_0^s D_\alpha u_r \delta B_r$ (use again Theorem 5 in [1]), and we get

$$\left| \left\langle \int_s^t Du_r \delta B_r, \gamma \mathbf{1}_{[s,t]} \right\rangle_{\mathcal{H}} \right| \leq C |t-s|^{\eta + \frac{m}{p}} Y_{m,p}^{\frac{1}{p}}, \quad (4.9)$$

where

$$Y_{m,p} = \int_0^1 \int_0^1 \int_0^1 \frac{|f_x^y D_\alpha u_r \delta B_r|^p}{|x-y|^{m+2}} dx dy d\alpha.$$

Substituting (4.5) and (4.9) into (4.4) yields

$$\begin{aligned} \langle \gamma D(X_t - X_s), 1_{[s,t]} \rangle_{\mathcal{H}} &\geq |t-s|^{\frac{2H(q+1)}{q}} Z_q^{-\frac{1}{q}} - C|t-s|^{\eta+\frac{m}{p}} Y_{m,p}^{\frac{1}{p}} \\ &= |t-s|^{\frac{2H(q+1)}{q}} \left(Z_q^{-\frac{1}{q}} - C|t-s|^{\delta} Y_{m,p}^{\frac{1}{p}} \right), \end{aligned}$$

where $\delta = \eta + \frac{m}{p} - 2H - \frac{2H}{q}$. With a right choice of η the exponent δ is positive, provided that $m - \frac{1}{p} - \frac{2H}{q} > 0$, because $\eta < 2H - \frac{1}{p}$. Taking into account that $\frac{2H}{q} < 1 - H$, it suffices that

$$m > \frac{1}{p} + 1 - H. \quad (4.10)$$

We construct now the sequence $\{F_n, n \geq 1\}$. Fix a natural number $n \geq 2$, and choose a function $\varphi_n(x)$, which is infinitely differentiable with compact support, such that $\varphi_n(x) = 1$ if $|x| \leq n-1$, and $\varphi_n(x) = 0$, if $|x| \geq n$. Set $F_n = \varphi_n(G)$, where $G = Z_q + Y_{m,p}$. Then clearly the sequences α_n and δ_n required in Theorem 1 can be constructed on the set $\{F_n \neq 0\}$, with $\theta = 2H + \frac{2H}{q}$.

It only remains to show that the random variables F_n are in the space $\mathbb{D}^{1,1}$. For this we have to show that the random variables $\|DZ_q\|_{\mathcal{H}}$ and $\|DY_{m,p}\|_{\mathcal{H}}$ are integrable on the set $\{G \leq n\}$. First notice that, as in the proof of Proposition 4.1 of [6], we can show that $E(\|DZ_q\|_{\mathcal{H}}) < \infty$. This follows from the integrability conditions I3) and

$$\int_0^1 E(\|Du_t\|_{\mathcal{H}}^p) dt < \infty, \quad (4.11)$$

which holds because of I2), the continuous embedding of $L^{\frac{1}{H}}([0,1])$ into \mathcal{H} (see [9]), and the fact that $pH \geq 1$. On the other hand, we can write

$$DY_{m,p} = p \int_0^1 \int_0^1 \int_0^1 |\xi_{x,y,\alpha}|^{p-1} \text{sign}(\xi_{x,y,\alpha}) D\xi_{x,y,\alpha} |x-y|^{-m-2} dx dy d\alpha,$$

where $\xi_{x,y,\alpha} = \int_y^x D_\alpha u_r \delta B_r$. Thus

$$\begin{aligned} \|DY_{m,p}\|_{\mathcal{H}} &\leq p \int_0^1 \int_0^1 \int_0^1 |\xi_{x,y,\alpha}|^{p-1} \|D\xi_{x,y,\alpha}\|_{\mathcal{H}} |x-y|^{-m-2} dx dy d\alpha \\ &\leq p(Y_{m,p})^{\frac{p-1}{p}} \left(\int_0^1 \int_0^1 \int_0^1 \|D\xi_{x,y,\alpha}\|_{\mathcal{H}}^p |x-y|^{-m-2} dx dy d\alpha \right)^{1/p}. \end{aligned}$$

Now, to show that $1_{(G \leq n)} \|DY_{m,p}\|_{\mathcal{H}}$ belongs to $L^1(\Omega)$, it suffices to show that the random variable

$$Y = \int_0^1 \int_0^1 \int_0^1 \|D\xi_{x,y,\alpha}\|_{\mathcal{H}}^p |x-y|^{-m-2} dx dy d\alpha$$

has a finite expectation. Since, for any $0 \leq y < x \leq 1$

$$D\xi_{x,y,\alpha} = \mathbf{1}_{[y,x]} D_\alpha u + \int_y^x DD_\alpha u_s \delta B_s^H,$$

we have

$$\begin{aligned} Y &\leq C \left(\int_0^1 \int_0^1 \int_0^1 \|\mathbf{1}_{[y,x]} D_\alpha u\|_{\mathcal{H}}^p |x-y|^{-m-2} dx dy d\alpha \right. \\ &\quad \left. + \int_0^1 \int_0^1 \int_0^1 \left\| \int_y^x DD_\alpha u_s \delta B_s^H \right\|_{\mathcal{H}}^p |x-y|^{-m-2} dx dy d\alpha \right) \\ &:= C(Y_1 + Y_2). \end{aligned}$$

From the continuous embedding of $L^{\frac{1}{H}}([0,1])$ into \mathcal{H} , we obtain

$$\begin{aligned} Y_1 &\leq C \int_0^1 \int_0^1 \int_0^1 \|\mathbf{1}_{[y,x]} D_\alpha u\|_{L^{1/H}([0,1])}^p |x-y|^{-m-2} dx dy d\alpha \\ &\leq C |x-y|^{pH-1} \int_0^1 \int_0^1 \int_0^1 \int_y^x |D_\alpha u_r|^p |x-y|^{-m-2} dr dx dy d\alpha. \end{aligned}$$

Hence, $E(Y_1) < \infty$, by Fubini's theorem, Proposition 3.1 in [6] and condition I2), provided

$$m < pH - 1. \quad (4.12)$$

On the other hand, using the estimate (2.2), and again the continuous embedding of $L^{\frac{1}{H}}([0,1])$ into \mathcal{H} yields

$$\begin{aligned} E \left(\left\| \int_y^x DD_\alpha u_s \delta B_s^H \right\|_{\mathcal{H}}^p \right) &\leq C E \left(\|D_\alpha Du \cdot \mathbf{1}_{[y,x]}(\cdot)\|_{\mathcal{H}^{\otimes 2}}^p + \|D_\alpha DDu \cdot \mathbf{1}_{[y,x]}(\cdot)\|_{\mathcal{H}^{\otimes 3}}^p \right) \\ &\leq C E \left(\| |D_\alpha Du \cdot| \mathbf{1}_{[y,x]}(\cdot) \|_{L^{1/H}([0,1]; \mathcal{H})}^p \right. \\ &\quad \left. + \| |D_\alpha DDu \cdot| \mathbf{1}_{[y,x]}(\cdot) \|_{L^{1/H}([0,1]; \mathcal{H}^{\otimes 2})}^p \right) \\ &\leq C |x-y|^{pH-1} \left(\int_y^x E \left(\| |D_\alpha Du_r| \|_{\mathcal{H}}^p \right) dr \right. \\ &\quad \left. + \int_y^x E \left(\| |D_\alpha DDu_r| \|_{\mathcal{H}^{\otimes 2}}^p \right) dr \right). \end{aligned}$$

As before we obtain $E(Y_2) < \infty$ by Fubini's theorem and condition I2), provided (4.12) holds. Notice that condition $\frac{1}{p} + 2 < H(p+1)$ implies that we can choose an m such that (4.10) and (4.12) hold.

Proof of condition b): Define $A_n = \{G \leq n\}$. Then, condition b) in Theorem 1 follows from

$$\begin{aligned} \int_T E(\langle \gamma DF_n, \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}}) |t-s|^{-\theta} dt ds &\leq C \int_T E(\mathbf{1}_{A_n} |\langle \gamma DG, \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}}|) |t-s|^{-\theta} dt ds \\ &\leq CE(\mathbf{1}_{A_n} \|DG\|_{\mathcal{H}}) \int_T |t-s|^{H-\theta} ds dt < \infty, \end{aligned}$$

since $E(\mathbf{1}_{A_n} \|DG\|_{\mathcal{H}}) < \infty$ and $\theta - H = H + \frac{2H}{q} < 1$.

Proof of condition c): We have

$$D_\alpha D_\beta (X_t - X_s) = \mathbf{1}_{[s,t]}(\beta) D_\alpha u_\beta + \mathbf{1}_{[s,t]}(\alpha) D_\beta u_\alpha + \int_s^t D_\alpha D_\beta u_r \delta B_r^H.$$

Hence

$$\begin{aligned} \left\langle \gamma^{\otimes 2} DD(X_t - X_s), \mathbf{1}_{[s,t]}^{\otimes 2} \right\rangle_{\mathcal{H}^{\otimes 2}} &= \left\langle \gamma^{\otimes 2} \mathbf{1}_{[s,t]}(\beta) D_\alpha u_\beta, \mathbf{1}_{[s,t]}^{\otimes 2} \right\rangle_{\mathcal{H}^{\otimes 2}} + \left\langle \gamma^{\otimes 2} \mathbf{1}_{[s,t]}(\alpha) D_\beta u_\alpha, \mathbf{1}_{[s,t]}^{\otimes 2} \right\rangle_{\mathcal{H}^{\otimes 2}} \\ &\quad + \left\langle \gamma^{\otimes 2} \int_s^t D_\alpha D_\beta u_r \delta B_r^H, \mathbf{1}_{[s,t]}^{\otimes 2} \right\rangle_{\mathcal{H}^{\otimes 2}} \\ &:= J_1(s, t) + J_2(s, t) + J_3(s, t). \end{aligned}$$

For $i = 1, 2, 3$, we set

$$A_i = E \left(F_n \int_T |t-s|^{-2\theta} |J_i(s, t)| ds dt \right).$$

Let us compute first

$$\begin{aligned} A_1 &\leq C \int_T \int_T |t-s|^{2H-2\theta} E \left(\|D_\alpha u_\beta| \mathbf{1}_{[s,t]}(\beta)\|_{\mathcal{H}^{\otimes 2}} \right) ds dt \\ &= C \int_T \int_T |t-s|^{2H-2\theta} \left(\int_s^t \int_s^t \varphi(\beta, y) d\beta dy \right)^{\frac{1}{2}} ds dt, \end{aligned}$$

where

$$\varphi(\beta, y) = \int_0^1 \int_0^1 E(|D_\alpha u_\beta| |D_x u_y|) \phi(\alpha, x) \phi(\beta, y) d\alpha dx.$$

By Fubini's theorem $A_1 < \infty$, because $2H - 2\theta > -2$, which is equivalent to $q > H$, and

$$\int_0^1 \int_0^1 \varphi(\beta, y) d\beta dy \leq E \left(\|Du\|_{\mathcal{H}^{\otimes 2}}^2 \right)$$

and this is finite because of the inclusion of $L^2([0, 1])$ in \mathcal{H} (4.3). In the same way we can show that $A_2 < \infty$. Finally,

$$\begin{aligned} A_3 &= E \left(F_n \int_T \int_T |t-s|^{-2\theta} \left| \left\langle \gamma^{\otimes 2} \int_s^t D_\alpha D_\beta u_r \delta B_r^H, \mathbf{1}_{[s,t]}^{\otimes 2} \right\rangle_{\mathcal{H}^{\otimes 2}} \right| ds dt \right) \\ &\leq C \int_T \int_T |t-s|^{2H-2\theta} E \left(\left\| \int_s^t DDu_r \delta B_r^H \right\|_{\mathcal{H}^{\otimes 2}} \right) ds dt, \end{aligned}$$

and we conclude as before by using for example the bound (2.2) for the norm of the Skorohod integral and the condition I2). \blacksquare

Remark 3 If $p = \frac{1+\sqrt{17}}{2}$, then $\frac{1}{p} + 2 < H(p+1)$ for all $H > \frac{1}{2}$.

References

- [1] Alòs, A. and Nualart, D. (2002). Stochastic integration with respect to the fractional Brownian motion. *Stochastics and Stochastics Reports* **75** 129–152.
- [2] Es-Sebaiy, K and Tudor, C.A. (2007): Multidimensional bifractional Brownian motion: Itô and Tanaka's formulas. *Stochastics and Dynamics* **3** 365–388.
- [3] Garsia, A. M., Rodemich, E. and Rumsey, H., Jr. (1970/1971). A real variable lemma and the continuity of paths of some Gaussian processes. *Indiana Univ. Math. J.* **20** 565–578.
- [4] Geman, D. and Horowitz, J. (1980). Occupation densities. *Ann. Probab.* **8** 1–67.
- [5] Houdré, C. and Villa, J. (2003). An example of infinite dimensional quasi-helix. *Contemporary Mathematics* **336** 195–201.
- [6] Imkeller, P. and Nualart, D. (1994). Integration by parts on Wiener Space and the Existence of Occupation Densities. *Ann. Probab.* **22** 469–493.
- [7] Kruk, I., Russo, F. and Tudor, C. A. (2007). Wiener integrals, Malliavin calculus and covariance structure measure. *J. Funct. Anal.* **249** 92–142.

- [8] Marcus, M. and Rosen, J. (2006). *Markov Processes, Gaussian processes and local times*. Cambridge University Press.
- [9] Mémin, J., Mishura, Y. and Valkeila, E. (2001) Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion. *Statist. Probab. Lett.* **51** 197–206.
- [10] Nualart, D. (2006). *The Malliavin calculus and related topics*. Springer Verlag. Second Edition.
- [11] Revuz, D. and Yor, M. (1994). *Continuous martingales and Brownian motion*. Springer Verlag.
- [12] Xiao, Y and Tudor C.A. (2007) Sample paths properties of the bifractional Brownian motion. *Bernoulli* **13**(4) 1023-1053.